Mathalon 2020 Round 1 (60 minutes MCQ)

Problem 1

Leaving at 9 a.m. this morning, Ryan decided to cycle back and forth to the top of the Virgin Rock. On the first part of the route the climb is light and he was able to drive at 18 km/h. On the second part, the slope increased and his speed dropped to 15 km/h. After reaching the top, he contemplated the beautiful landscape for a quarter of an hour and then turned around. He went down the first steep incline at a speed of 30 km/h first and then finished at 22.5 km/h on the least inclined part of the trajectory. His hike ended at 11:30 am.

How far did Ryan travel this morning in total?

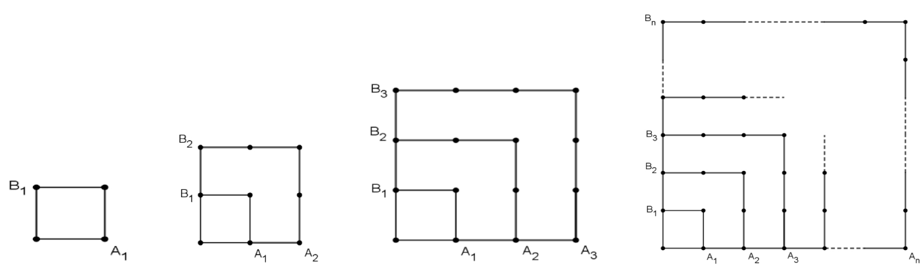
Problem 2

We consider a sequence of networks built on a square mesh of side 1, as described below:

R_1 : 4 vertices and 4 edges of length 1

R_2 : 9 vertices and 10 edges of length 1

R_3 : 16 vertices and 18 edges of length 1



R_1 R_2 R_3 R_n

Each network is made up of vertices, and edges of length 1.

A robot is placed on one of the vertices of the network. It moves from vertex to vertex, following the edges of the network.

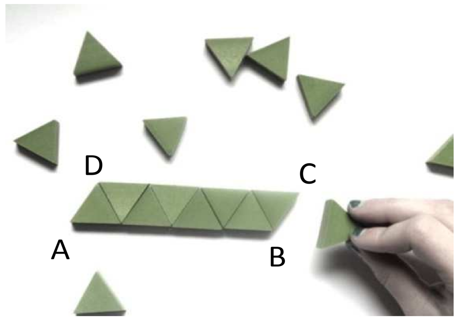
We assume the robot is programmed to stop after having traversed all the edges of the network. The robot's path is determined by its initial point, which can be any vertex in the network, and its terminal after passing at least once through all edges of the network. The length of the route is the number of edges of length 1 of the robot path.

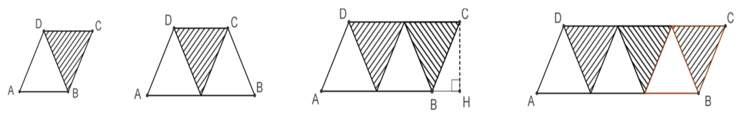
We propose to minimize the length of the robot path on the R_n network by specifying the initial point of on the network, the shortest possible path through all edges of length 1 of the network.

The length of this minimum path is then denoted by L_n. What is L_3?

Problem 3

Patiently, Joe aligns identical equilateral triangles of its mosaic set by juxtaposing them as shown below. His sister, Amy, who is always in search of a few calculations to do, has fun finding the exact value the lengths of the diagonals of the quadrilaterals obtained. Each equilateral triangle has side 1. We note : ABCD a quadrilateral built by Joe ;
L = AC the length of the diagonal [AC];

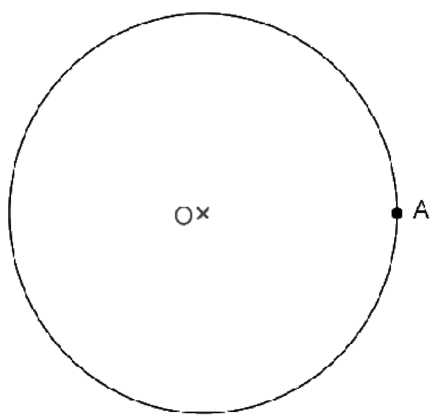




If Joe aligns 96 triangles, what is the value of L?

Problem 4

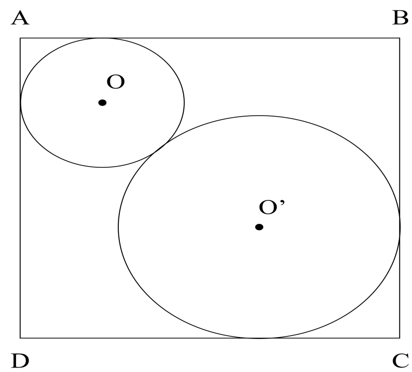
Let $\mathcal{C}$ be the circle centered a O and A a point on the circle. Denote by $\mathcal{D}$ the disk with boundary $\mathcal{C}$.



Let M be an equiprobably randomly chosen point on the disk surface. What is the probability that M is closer to O than to A?

Problem 5

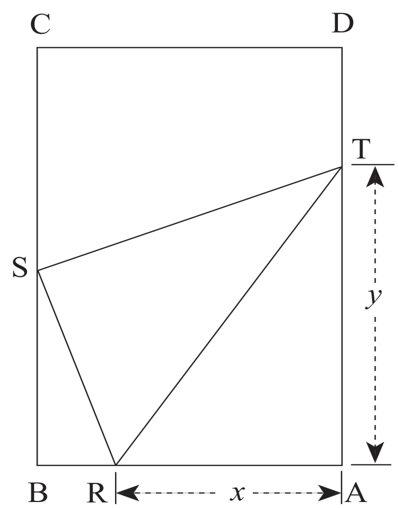
Consider a square ABCD of side a. A circle Γ inside the square is tangent to (AB) and (AD). A second circle Γ', inside the square, is tangent to Γ and (CB) and (CD).
Let S be the sum of the areas of the circles Γ and Γ'.



Denote by S_{min} the minimum value of S and S_{max} the maximum value of S. Find $S_{min}+S_{max}$.

Problem 6

Let ABCD be a rectangular sheet having a width $AB = 4$ and length $BC = 6$. Let R a point in the line segment [AB] (lower edge of the sheet) and T a point in the line segment [AD] (right edge of the sheet). The sheet is folded along the line segment [RT]. The new position of A is denoted by S is a point on the line segment [CB] as shown in the figure below.



Put $AR = x$ and $AT = y$. Denote α and β the minimum and maximum values of x respectively. Find $\beta - \alpha$

Problem 7
Let $a,\ b,\ c,$ and d be four real numbers such that

$$a < b < c < d.$$

If

$$x = (a+b)(c+d), \quad y = (a+c)(b+d), \text{ and } z = (a+d)(b+c)$$

Which of the following is TRUE?

(a) $x < y < z$

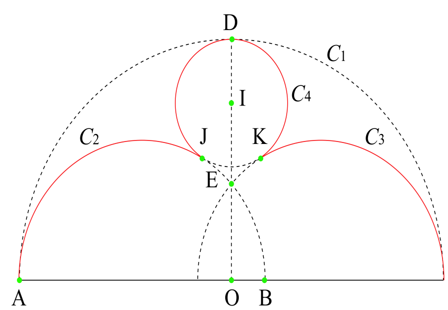
(b) $y < x < z$

(c) $x < z < y$

(d) $x - y > \frac{1}{2}(z - y)$

(e) $y - z > \frac{1}{3}(y - x)$

Problem 8
For the purposes of his new show, a famous singer wants to create a modern stage show.The scene is shown from above by the following figure:



The semi-circle $\mathcal{C}_1$ with center O passing through the point A and the semi-circle $\mathcal{C}_2$ with diameter $[AB]$ are tangent at A. The line (OD) is an axis of symmetry of the figure and the point D is on $\mathcal{C}_1$. The semi-circle $\mathcal{C}_3$ is symmetric to $\mathcal{C}_2$ with respect to (OD). The line segment $[OD]$ and $\mathcal{C}_2$ intersect at E. $\mathcal{C}_4$ is the circle centered at I passing through the point D. $\mathcal{C}_4$ is tangent to $\mathcal{C}_1$ at D, tangent to $\mathcal{C}_2$ at J, and tangent to $\mathcal{C}_3$ at K.
Construction constraints impose that

$$OA = 10\ m \quad \text{and} \quad DE = 6\ m.$$

Denote by R_2 and R_4 the radii of $\mathcal{C}_2$ and $\mathcal{C}_4$ respectively. Find $R_2 + R_4$.

Problem 9

Let $g : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$g(m + f(n)) = n + g(m + 95) \qquad \forall n, m > 0.$$

Find $\displaystyle\sum_{k=1}^{19} g(k)$.

Problem 10

At an international mathematics conference, delegations from the three countries France, Belgium and Canada each arrive in a different minibus. When arriving at the university parking lot, mathematicians of different nationality greet each other by exchanging kisses. But the custom is different in each country: the French are used to making two kisses, the Belgians make one and the Canadians make three. When two people meet, it is the number of kisses of the one who makes it the most that is exchanged.

In all, 648 kisses are exchanged, and there are 27 mathematicians. One of them points out that Canadians are twice as numerous as Belgians.

Determine the number of Belgian mathematicians who attended the conference.

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Problem 1

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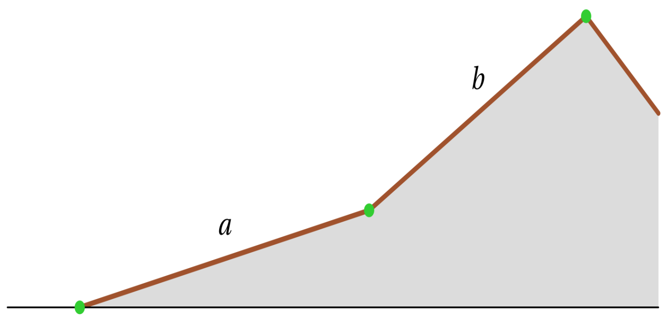
How far did Ryan travel this morning in total?

Solution



Ryan' journey, on the way back and forth, consists of two parts, lengths a and b.

$$\frac{a}{18}+\frac{b}{15}+\frac{1}{4}+\frac{b}{30}+\frac{a}{22.5}=\frac{5}{2}$$

$$a+b=\frac{45}{2}$$

The distance traveled by Ryan is therefore $2\,(a+b)$, i.e. 45km.

$$Ans = 45km$$

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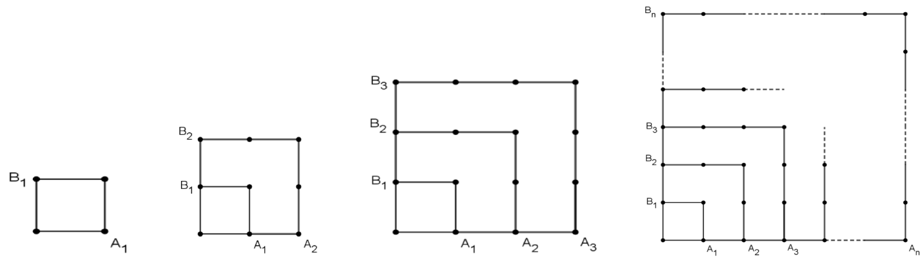

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A robot is placed on one of the vertices of the network. It moves from vertex to vertex, following the edges of the network.
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We propose to minimize the length of the robot path on the R_n network by specifying the initial point of on the network, the shortest possible path through all edges of length 1 of the network.
The length of this minimum path is then denoted by L_n. What is L_3?

Solution

The R_1 network consists of 4 edges of length 1, i.e. $L_1 \geq 4$. The robot can traverse this network starting from B_1 and turning around counterclockwise. So $L_1 = 4$.
The R_2 network has 10 edges of length 1 so $L_2 \geq 10$. The robot can traverse this network starting from B_1 and turning around the R_1 network ounterclockwise until you get back to B_1 and then by going along the outer edge passing through B_2, then A_2 and finally A_1. So $L_2 = 10$.
The R_3 network consists of 18 edges of length 1, i.e. $L_3 \geq 18$. If the robot could travel the entire network with a route length of 18, then the number of edges starting from each vertex would be even, except perhaps for two of them: the first and the last of the chain. But here exactly three edges resulted from the four vertices B_1, B_2, A_1 and A_2. It is therefore not possible to traverse the network with a path length of 18. It is thus deduced that $L_3 \geq 19$.
Moreover, the robot can traverse this network by successively performing :
- the path of network R_2 from B_1 to A_1 of length 10 ($L_2 = 10$)
- a backward step of length 1 from A_1 to A_2
- a course of the remaining outer edge of length 8 (from A_2 to B_2 through A_3 then B_3).

The proposed route has length 19 and $L_3 \geq 19$ so $L_3 = 19$.

$$Ans : L_3 = 19.$$

Problem 3

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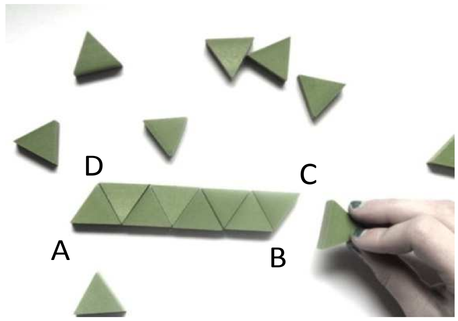
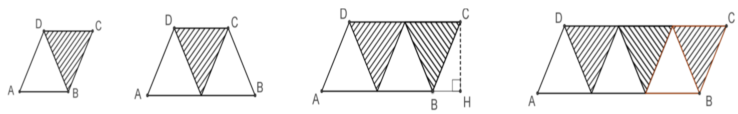
L = AC the length of the diagonal [AC];





If Joe aligns 96 triangles, what is the value of L?

Solution

When the number n of triangles is even, $n = 2p$, the largest of the diagonals is the hypotenuse. of a right triangle with one side of the right angle having a length of $p + \frac{1}{2}$ and the other $\frac{\sqrt{3}}{2}$. By using pythagoras

$$L = \sqrt{p^2 + p + 1}$$

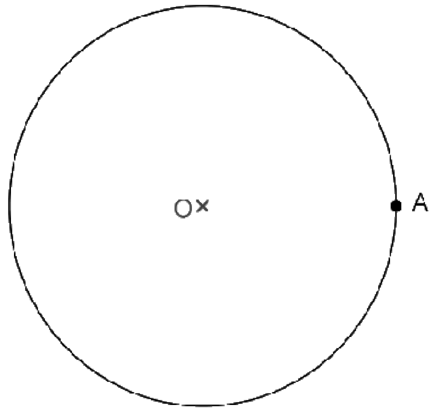
$$96 = 2\,(48)$$

$$L = \sqrt{(48)^2 + 48 + 1} = \sqrt{2353} = 48.\,508$$

$$Ans = 48.\,508$$

Problem 4

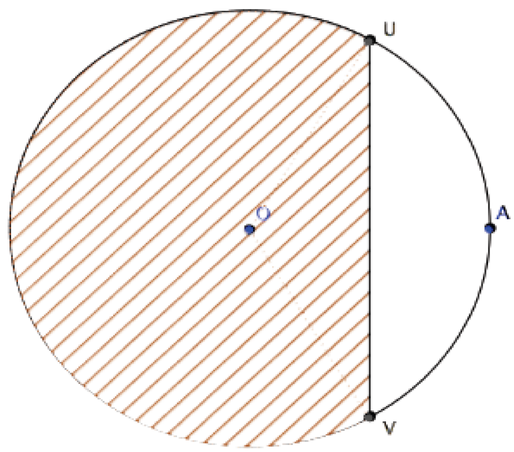
Let $\mathcal{C}$ be the circle centered a O and A a point on the circle. Denote by $\mathcal{D}$ the disk with boundary $\mathcal{C}$.



Let M be an equiprobably randomly chosen point on the disk surface. What is the probability that M is closer to O than to A?

Solution

The probability that M is closer to O than to A is the ratio of the shaded area to that of the disk. The area of the disk (of radius R) : is πR^2 .



The UOV angular sector has an area one third the size of the previous one. The area of the shaded portion is the sum of two thirds of the area of the disc and the area of the OUV triangle: $\frac{2}{3}\pi R^2 + \frac{1}{4}\sqrt{3}R^2$.

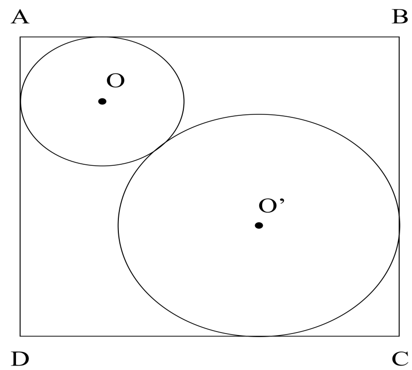
The probability that M is closer to O than to A is

$$\frac{\frac{2}{3}\pi R^2 + \frac{1}{4}\sqrt{3}R^2}{\pi R^2} = 0.804\,50$$

$$Ans = 0.804\,50$$

Problem 5

Consider a square ABCD of side a. A circle Γ inside the square is tangent to (AB) and (AD). A second circle Γ', inside the square, is tangent to Γ and (CB) and (CD).
Let S be the sum of the areas of the circles Γ and Γ'.



Denote by S_{min} the minimum value of S and S_{max} the maximum value of S. Find $S_{min}+S_{max}$.

Solution

The centers O and O' of the circles being at equal distance from the sides AB and AD for one and on the CB and CD sides for the other, the centers of the two circles are located on the AC diagonal and the radii r and r' of the circles satisfy

$$OA + r + r' + OC = a\sqrt{2}$$
$$(r + r')\left(1 + \sqrt{2}\right) = a\sqrt{2}.$$

Thus

$$r + r' = a\left(2 - \sqrt{2}\right)$$

The circles being located inside a square of side a, their radii remain smaller than $\frac{a}{2}$. We deduce that each radius belongs to the interval

$$\left[a\left(\frac{3}{2} - \sqrt{2}\right),\ \frac{a}{2}\right].$$

The sum of the areas of the two circles is

$$\begin{aligned} S &= \pi\left(r^2 + r'^{\,2}\right) \\ &= \frac{\pi}{2}\left[(r + r')^2 + (r - r')^2\right] \\ &= \frac{\pi}{2}\left[\left(6 - 4\sqrt{2}\right)a^2 + (r - r')^2\right] \end{aligned}$$

It is immediately deduced that this area is minimal when

$$r = r' = a\left(1 - \frac{\sqrt{2}}{2}\right).$$

Hence,

$$S_{\min} = \pi \left(3 - 2\sqrt{2}\right) a^2.$$

S is maximal when r is maximal and r' minimal (or conversely) that is when

$$r = \frac{a}{2} \quad \text{and} \quad r' = a\left(\frac{3}{2} - \sqrt{2}\right).$$

Hence,

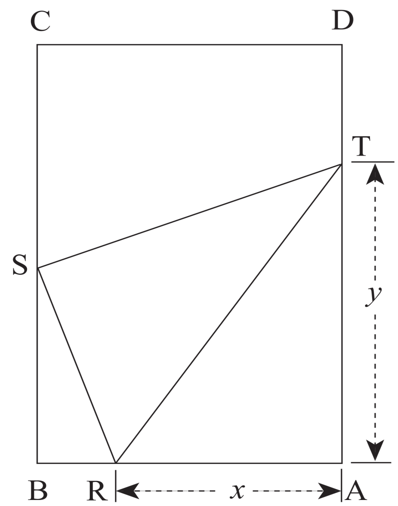
$$\begin{aligned} S_{\max} &= \frac{\pi}{2}\left[\left(6 - 4\sqrt{2}\right) a^2 + \left(-1 + \sqrt{2}\right)^2 a^2\right] \\ &= \pi\left(\frac{9}{2} - 3\sqrt{2}\right) a^2 \end{aligned}$$

$$S_{\min} + S_{\max} = \pi\left(3 - 2\sqrt{2}\right) a^2 + \pi\left(\frac{9}{2} - 3\sqrt{2}\right) a^2 \simeq 0.43\pi a^2$$

$$Ans = 0.43\pi a^2$$

Problem 6

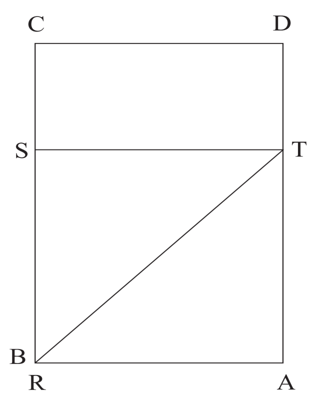
Let ABCD be a rectangular sheet having a width $AB = 4$ and length $BC = 6$. Let R a point in the line segment [AB] (lower edge of the sheet) and T a point in the line segment [AD] (right edge of the sheet). The sheet is folded along the line segment [RT]. The new position of A is denoted by S is a point on the line segment [CB] as shown in the figure below.



Put $AR = x$ and $AT = y$. Denote α and β the minimum and maximum values of x respectively. Find $\beta - \alpha$

Solution

The maximum value of x is achieved when R is at B.

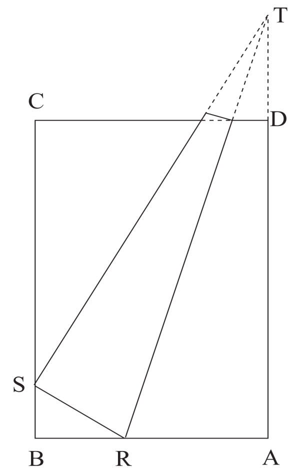


In this case $x = AB = 4$

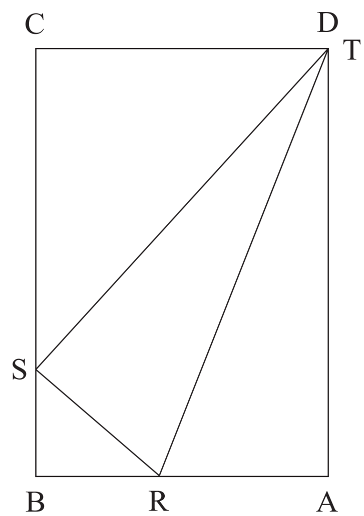
After the folding, the new position of A, namely S is a point in the line segment [BC] only if

$$RA \geq RB \Leftrightarrow x \geq \frac{a}{2}. \quad (a = 4)$$

But for the smaller values of x larger than $\frac{a}{2}$, the point T which is the intersection of the folding line and (AD) will be outside of the line segment [AD]. This is shown in the figure below



The folded part is then a trapezoid.
The smallest value of x for which the problem will make sense will be obtained when T is at D as shown below.



Thus,

$$DS = DA = 6.$$

Since $CD = 4,\ CS = \sqrt{20}$ by pythagoras, we have

$$BS = 6 - \sqrt{20}.$$

Using $BR = 4 - x$, we get

$$RS^2 = AR^2 = x^2 = \left(6 - \sqrt{20}\right)^2 + (4 - x)^2 \text{ by Pythagoras,}$$

then

$$\left(6 - \sqrt{20}\right)^2 + 16 - 8x = 0 \Leftrightarrow 8x = 72 - 12\sqrt{20}.$$

So,

$$x = 9 - \frac{3}{2}\sqrt{20}.$$

$$x \in \left[9 - \frac{3}{2}\sqrt{20},\ 4\right]$$

$$\alpha = 9 - \frac{3}{2}\sqrt{20} \text{ and } \beta = 4$$

$$\beta - \alpha = 4 - \left(9 - \frac{3}{2}\sqrt{20}\right) = 3\sqrt{5} - 5 = 1.7082 \simeq 1.7$$

$$Ans = 1.7$$

Problem 7
Let $a,\ b,\ c,$ and d be four real numbers such that

$$a < b < c < d.$$

If

$$x = (a+b)(c+d), \quad y = (a+c)(b+d), \text{ and } z = (a+d)(b+c)$$

Which of the following is TRUE?

(a) $x < y < z$

(b) $y < x < z$

(c) $x < z < y$

(d) $x - y > \frac{1}{2}(z - y)$

(e) $y - z > \frac{1}{3}(y - x)$

Solution
On one hand,

$$\begin{aligned} x - y &= ac + bc + ad + bd - ab - ad - cb - cd \\ &= a(c-b) + d(b-c) = (d-a)(b-c) < 0 \end{aligned}$$

On the other hand

$$\begin{aligned} y - z &= ab + ad + cb + cd - ab - ac - db - dc \\ &= a(d-c) + b(c-d) = (a-b)(d-c) < 0 \end{aligned}$$

Hence,

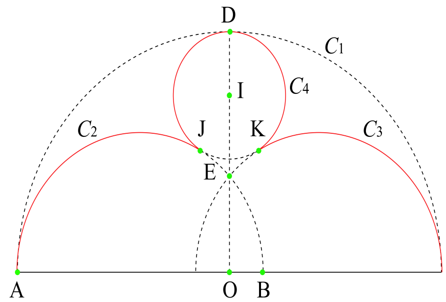
$$x < y < z$$

The answer is (a) only

$$Ans: \text{ (a) only}$$

Problem 8

For the purposes of his new show, a famous singer wants to create a modern stage show.The scene is shown from above by the following figure:



The semi-circle $\mathcal{C}_1$ with center O passing through the point A and the semi-circle $\mathcal{C}_2$ with diameter $[AB]$ are tangent at A. The line (OD) is an axis of symmetry of the figure and the point D is on $\mathcal{C}_1$. The semi-circle $\mathcal{C}_3$ is symmetric to $\mathcal{C}_2$ with respect to (OD). The line segment $[OD]$ and $\mathcal{C}_2$ intersect at E. $\mathcal{C}_4$ is the circle centered at I passing through the point D. $\mathcal{C}_4$ is tangent to $\mathcal{C}_1$ at D, tangent to $\mathcal{C}_2$ at J , and tangent to $\mathcal{C}_3$ at K.
Construction constraints impose that

$$OA = 10\ m \quad \text{and} \quad DE = 6\ m.$$

Denote by R_2 and R_4 the radii of $\mathcal{C}_2$ and $\mathcal{C}_4$ respectively. Find $R_2 + R_4$.

Solution

Note that if two circles centered at O and O′ are tangent at M, then O, O′ and M will be on the same line.

$$E \in [OD] \Rightarrow OE = OD - ED = 10 - 6 = 4.$$

In the right triangle AOE, we have

$$AE^2 = AO^2 + OE^2$$

$$AE^2 = 10^2 + 4^2 \Rightarrow AE = \sqrt{116}$$

$$E \in \mathcal{C}_2 \Rightarrow ABE \text{ is a right triangle at } E$$

AOE and ABE are right triangles that have an acute angle in common. Thus,

$$\frac{AO}{AE} = \frac{AE}{AB} \Rightarrow AB = \frac{AE^2}{AO} = 11.6. \text{ So } R_2 = \frac{11.6}{2} = 5.8$$

Now if we denote by Ω the center of the circle $\mathcal{C}_2$ and R_4 the radius of $\mathcal{C}_4$, then

$$\Omega I = 5.8 + R_4$$

and

$$I \in [OD] \Rightarrow OI = OD - ID = 10 - R_4,$$

$$\Omega \in [AO] \Rightarrow \Omega O = AO - A\Omega = 10 - 5.8 = 4.2.$$

In the right triangle ΩOI (at O), we have

$$\Omega I^2 = \Omega O^2 + OI^2$$

$$(5.8 + R_4)^2 = 4.2^2 + (10 - R_4)^2 .$$

After expanding and solving for R_4, we get

$$R_4 = \frac{210}{79} = 2.6582$$

$$R_2 + R_4 = 5.8 + 2.6582 = 8.4582 \simeq 8.46$$

$$Ans = 8.46$$

Problem 9
Let $g : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$g(m + f(n)) = n + g(m + 95) \qquad \forall n, m > 0.$$

Find $\sum_{k=1}^{19} g(k)$.

Solution
Put

$$G(n) = g(n) - 95 \text{ and } m + 95 = k.$$

Then for $n \geq 1$ and $k \geq 96$, we have

$$G(k + G(n)) = n + G(k). \qquad (1)$$

It follows that

$$G(k + G(k + G(n))) = G(k + n + G(k))$$

According to (1), we have in one hand

$$G(k + G(k + G(n))) = k + G(n) + G(k)$$

on the other hand

$$G(k + n + G(k)) = k + G(k + n).$$

Thus,

$$G(k + n) = G(n) + G(k) \text{ for all } n \geq 1 \text{ and } k \geq 96. \qquad (2)$$

We deduce by induction that

$$G(p) \geq pG(1) \text{ for all } p \geq 1.$$

Indeed, this is true when $p = 1$. Suppose it is true for a fixed $p \geq 1$. By virtue of (2) and the induction hypothesis, we have

$$\begin{aligned} G(p + 1 + 96) &= G(97) + G(p) = G(96) + G(1) + G(p) \\ &= G(96) + (p + 1)G(1) \end{aligned}$$

Again by (2),
we have

$$G(p + 1 + 96) = G(p + 1) + G(96).$$

Now by comparing the two equalities, we obtain

$$G(p + 1) = (p + 1)\, G(1).$$

Next, for $n \geq 1$ and $k \geq 96$, we then have

$$G(k + G(n)) = (k + G(n))\, G(1) = (k + nG(1))\, G(1) = kG(1) + n\,(G(1))^2$$

and, according to (1)

$$G\left(k+G(n)\right)=n+G(k)=n+kG(1)\Rightarrow\left(G(1)\right)^2=1\Rightarrow G(1)=1.$$

Thus, for all $n\geq 1$

$$G(n)=n\Leftrightarrow g(n)=n+95.$$

$$\sum_{k=1}^{19}g(k)=\sum_{k=1}^{19}(k+95)=1995.$$

$$Ans=1995$$

Problem 10

At an international mathematics conference, delegations from the three countries France, Belgium and Canada each arrive in a different minibus. When arriving at the university parking lot, mathematicians of different nationality greet each other by exchanging kisses. But the custom is different in each country: the French are used to making two kisses, the Belgians make one and the Canadians make three. When two people meet, it is the number of kisses of the one who makes it the most that is exchanged.

In all, 648 kisses are exchanged, and there are 27 mathematicians. One of them points out that Canadians are twice as numerous as Belgians.

Determine the number of Belgian mathematicians who attended the conference.

Solution

Denote by f the number of French, b the number of Belgians and c the number of Canadians. The number N of kisses exchanged is given by:

$$N = 2bf + 3bc + 3fc.$$

The data of the problem make it possible to write the following system of 3 equations with 3 unknowns:

$$\begin{cases} 2bf + 3bc + 3fc = 648 \\ f + b + c = 27 \\ c = 2b \end{cases}$$

By substituting, we obtain

$$-18b^2 + 216b - 648 = 0$$

$\Leftrightarrow$

$$-b^2 + 12b - 36 = 0 \Rightarrow b = 6$$

$$Ans = 6.$$

Mathalon 2020 Round 2 (3 minutes)

Problem 1
Express the following in terms of $\cos x$ or $\sin x$.

$$\sin\left(x+\frac{\pi}{2}\right)+\sin(x+\pi)+\cos\left(\frac{\pi}{2}-x\right)$$

$$Ans : \cos x$$

Problem 2
What is the equation of the tangent line to the graph of $y=\sqrt{2x+5}$ at $x=2$?

$$Ans : \; y=\frac{1}{3}x+\frac{7}{3}$$

Problem 3
If u_n is an arithmetic sequence with $u_2=10$ and $u_3=20$, what is u_1?

$$Ans : u_1=0$$

Problem 4
If z is a complex number, what is the solution of the equation $\; 2+4z=5\bar{z}-63i$?

$$Ans : \; z=2-7i$$

Problem 5
What is the modulus of the complex number $\dfrac{1}{(1-i)^{10}}$?

$$Ans : \; \frac{1}{32}$$

Problem 6
Give an antiderivative of $f(x)=5e^{3-2x}$

$$Ans : \frac{-5}{2}e^{3-2x}$$

Problem 7
What is the value of the integral

$$\int_{-\pi}^{\pi}\left(e^{x^2}\sin^5 x+\frac{x^3}{5}-\cos x+1\right)dx?$$

$$Ans : 2\pi$$

Problem 8
What is the average value of $f(x)=|x|$ on the interval $[-3,\; 3]$?

$$Ans : \; 1.5$$

Problem 9

What is

$$\frac{d^{(999)}}{dx^{999}}(\sin x)?$$

$$Ans: -\cos x$$

Problem 10
What is the average value of $f(x) = 2\sqrt{1-x^2}$ on the interval $[0, 1]$?

$$\int_0^1 2\sqrt{1-x^2}dx$$

$$Ans: \frac{\pi}{2}$$

Mathalon 2020 Round 3 (20 minutes)

Problem 1
Find all the functions $g : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\frac{1}{x}g(-x) + g(\frac{1}{x}) = x, \quad \text{for all } x \neq 0.$$

Solution
Let g be such function and a an arbitrary nonzero real number.

- Put $x = -a$ and multiply both sides of the equality by a $(a \neq 0)$. We obtain

$$g(a) - ag(\frac{-1}{a}) = a^2 \qquad (1)$$

- Next, put $x = \dfrac{1}{a}$ to obtain

$$g(\frac{1}{a}) - \frac{1}{a}g(-a) = \frac{1}{a^2}$$

$\Leftrightarrow$

$$ag(\frac{1}{a}) - g(-a) = \frac{1}{a}.$$

Consequently by substituting $-a$ for a, we get

$$-ag(\frac{-1}{a}) - g(a) = \frac{-1}{a} \qquad (2)$$

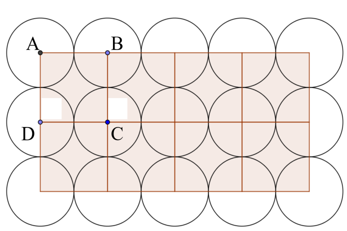
$(1) - (2)$ gives

$$2g(a) = a^2 + \frac{1}{a}$$

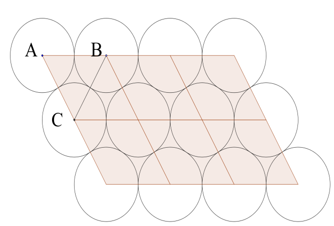
$$g(a) = \frac{1}{2}\left(a^2 + \frac{1}{a}\right) = \frac{a^3+1}{2a}$$

$g(x) = \dfrac{x^3+1}{2x}$ is the function that we are looking for.

Problem 2
A tiler has a (sufficient) stock of blue and round pavers, 10 cm radius, to pave a large room. He hesitates between the following two types of paving :



Paving P_1



Paving P_2

On each paving, when connecting the centers of the adjacent discs, we obtain a polygon (called cell) that reproduces "to Infinity". In the first paving this cell is a square and in the second this cell is an equilateral triangle.
The paving density is defined to be the ratio of the area occupied by the disk portions contained in a cell and the surface of the cell itself. This defines an indicator compactness of paving: the greater the density of paving, the more compact the paving.T.
Find the paving densities of P_1 and P_2. Compare the two pavings. Which one is the best?

Solution
Consider the square ABCD of P_1 (the areas are expressed in cm2). This square has a side of 20 cm and its interior contains 4 quarters of disk radius 10 cm. We have

$$D_1 = \frac{4\left(\frac{\pi 10^2}{4}\right)}{20^2} = \frac{\pi}{4} = 0.785\,40.$$

Let's now consider the equilateral triangle ABC of P_2. This triangle has 20 cm of side and its interior contains three disk portions each corresponding to an angular sector with angle $\frac{\pi}{3}$. This gives

$$A_s = 3 \times \frac{\pi}{6} \times 10^2 = 50\pi.$$

On the other hand, the area of the triangle is

$$A_T = \frac{b \times h}{2} = \frac{(20)\left(10\sqrt{3}\right)}{2} = 100\sqrt{3}$$

Hence,

$$D_2 = \frac{50\pi}{100\sqrt{3}} = 0.906\,90.$$

We observe that $D_2 > D_1$. Therefore, P_2 is the best paving.